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Discrete Applied Mathematics 148 (2005) 263–272

DISCRETE APPLIED MATHEMATICS

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Note

A note on rectilinear and polar visibility graphs ☆

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Received 13 November 2003; received in revised form 27 November 2004; accepted 15 December 2004
Available online 4 May 2005

Dedicated to the memory of J. Philip Huneke

Abstract

We give a new, inductive proof that every 2-connected planar graph is a bar-visibility graph. Changing from horizontal lines to arcs of concentric circles and from vertical to radial visibility, we obtain a similar result for polar visibility graphs, which naturally embed on the projective plane.

Keywords: Bar-visibility graph; Planar graph; Projective planar graph

1. Introduction

It is well known that every 2-connected planar graph has a bar-visibility representation, that is, a representation in the plane with vertices represented by disjoint, horizontal line segments and with edges represented by vertical visibility between segments. Even more, a planar graph has a bar-visibility layout if and only if it can be drawn in the plane with all cut-vertices on a common face. Here we give a new (inductive) proof of the former fact using ideas related to those of [10,15]. The usual proof [12,14,18] first applies the result that a 2-connected graph has an st-numbering [5], and then uses this numbering to complete the bar-visibility proof and to construct a layout for such graphs.

☆ Research supported in part by N.S.A. Grant #MDA904-99-1-0069 and by Membership at MSRI, Berkeley, CA, fall, 2000.

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doi:10.1016/j.dam.2004.12.004

We also apply this proof technique for an extension to graphs laid out using polar coordinates or with polar visibility, that is, with vertices represented by arcs of concentric circles and edges by radial visibility, outward and through the origin. In this context as well, it is not hard to see that all 2-connected graphs that embed in the plane or on the real projective plane (the nonorientable surface of Euler characteristic 1) have a polar visibility layout. In [6,7] it is shown in a more complex proof that a graph is a polar visibility graph (PVG) if and only if it embeds in the plane with all but at most one cut-vertex on a common face or on the projective plane with all cut-vertices on a common face. The newer inductive proof leads to a recursive algorithm, though it seems to be less efficient than the usual linear-time algorithms of [12,14,18]. Similar visibility layouts on the Möbius band have been characterized in [3].

The goal is to provide a simple proof of the following.

Theorem 1. *A* 2-*connected graph that embeds in the plane* (*respectively*, *the projective plane*) *has a bar-visibility* (*resp.*, *polar visibility*) *representation.*

First we present the necessary and common background for the two results. In Section 3 we prove a stronger theorem for planar graphs and in Section 4 one for projective planar graphs, explaining the variations needed for the latter graphs. We conclude with a few thoughts on the related algorithmic problem of finding these visibility layouts.

2. Background

A *bar-visibility graph* (*BVG*) is one whose vertices can each be represented by a closed horizontal line segment in the plane, with segments disjoint and lying on distinct horizontal lines, and with two vertices adjacent in the graph if and only if the corresponding segments are vertically visible. Two segments are *vertically visible* if there is a nondegenerate rectangle that intersects only these two segments and whose horizontal sides are subsets of these segments. A bar-visibility representation of $K_{2,3}$ is shown in Fig. 1.

Similarly a polar visibility graph (*PVG*) is one whose vertices are each represented by an arc with the arcs closed, proper subarcs of circles centered at the origin, disjoint and lying on distinct circles, so that two vertices are adjacent if and only if the corresponding arcs are radially visible (to be defined below). We define a (nondegenerate) *cone* in the plane to be a 4-sided region of positive area with two opposite sides being subarcs of circles, centered at the origin, and the other two sides, possibly intersecting, being radial line segments on lines

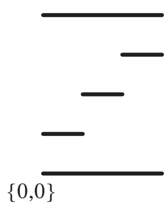

Fig. 1. $K_{2,3}$ as a BVG.

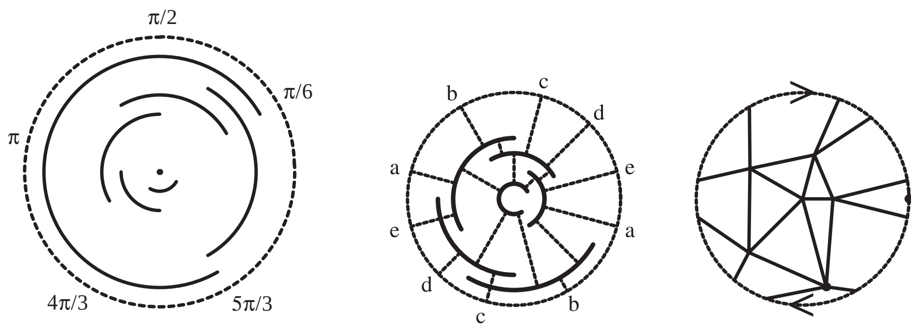

Fig. 2. (a) A layout L representing K_6 as a PVG; (b) $i(L)$ and $i(L)^*$; (c) G_L on the projective plane.

through the origin. Thus, both $\{(r,\theta) : 1\leqslant r\leqslant 2;\ 0\leqslant\theta\leqslant\pi/6\}$ and also

$$\{(r,\theta) : 0\leqslant r\leqslant 1;\ 0\leqslant\theta\leqslant\pi/6 \ \mathit{or}\ \ \pi\leqslant\theta\leqslant 7\pi/6\}$$
$$= (r,\theta) : -1\leqslant r\leqslant 1;\ 0\leqslant\theta\leqslant\pi/6\}$$

are considered to be cones, respectively, not containing and containing the origin. Two arcs are said to be *radially visible* if there is a cone that intersects only these two arcs and whose two circular ends are subsets of the two arcs. Fig. 2a gives a polar visibility layout of K_6. In this work, by arc we always mean a proper subarc of a circle. Polar layouts in which full circles as well as arcs are used (*CVGs*) are characterized in [6,7] and are shown to include a wider class of planar and projective planar graphs. For BVGs and PVGs, we refer to the rectangle or cone of visibility between two segments (bars or arcs) as *visibility bands*. Different points of view, but using circular arcs, appear in [1,2].

We need some basic results from graph theory [16] and on embedded graphs [11,17]. G is 2-*connected* if G has at least three vertices and the removal of no single vertex and its incident edges disconnects the graph, i.e., G has no *cut-vertex*. Note that K_1 and K_2 have no cut-vertex, but are not 2-connected. A *cut-edge* is an edge whose removal leaves a disconnected graph. A *block* is a maximal connected subgraph that contains no cut-vertex. Then the blocks of a connected graph are its cut-edges and its maximal 2-connected subgraphs.

For G simple, the *contraction* of an edge $e=\{u,v\}$ produces the simple graph G/e, the graph obtained by identifying u and v to become a new vertex adjacent to all vertices previously adjacent to u or v, removing e, and replacing any set of multiple edges by a single edge. An edge $e=\{u,v\}$ is said to be *simple* (respectively *multiple*) if vertices u and v are not (resp., are) joined by an additional edge. A graph is said to be *plane* (respectively, *projective plane*) if it is planar and embedded in the plane (resp., if it can be and is embedded on the projective plane). Recall that the (real) projective plane can be modeled by taking a circular disk and identifying opposite (or antipodal) points, as shown in Fig. 2b,c. For a multigraph G embedded on any surface and e a nonloop edge, G/e is the embedded multigraph obtained by contracting e on the surface to become a new vertex and removing e.

The class of plane graphs arising naturally from BVGs consists of the loopless planar graphs embedded with no finite digon (or 2-sided) face, though the infinite, exterior face can be a digon. For G loopless, plane with no finite digon face, and $e=\{u,v\}$ a simple edge,

G/e is the loopless, plane graph with no finite digon face obtained by contracting e to a new vertex, removing e, and replacing each finite digon face by a single edge. Note that the contraction of an edge of a loopless plane multigraph leaves the graph loopless if and only if the edge is simple. Also a finite digon face is formed precisely when e is incident with one or two triangular faces in G, but then each is replaced by a single edge. In this class of graphs there are two types of multiple edges—it might be that a pair of multiple edges bounds the infinite face; otherwise at least one of the pair lies in the interior, not on the infinite face. In a 2-connected plane graph with no finite digon face, every pair of multiple edges must have nonempty interior, a block adjacent to both end-vertices.

What is an embedded graph? If a graph is embedded on a surface, then for each vertex there is naturally defined a local cyclic ordering of its incident edges, given by the order, say clockwise, of its edges in the embedding; such a collection of cyclic rotations, one for each vertex, is called a *rotation system*. (See, for example, [11,17] where it is shown that an embedding on an orientable surface is equivalent to a rotation system.) Two graphs embedded on the same orientable surface are said to be *equivalent* if at each vertex the corresponding rotations agree; such graphs are necessarily isomorphic as abstract graphs. Note that in the context of planar multigraphs embedded without finite digon faces, a neighboring vertex may appear twice or more on different edges in a rotation, but not consecutively unless the two edges to that vertex bound the infinite face. Similarly, given a bar-visibility layout (respectively, polar visibility layout) L in the plane, one can define the *bar-rotation system* (resp., the *arc-rotation system*) to be the set of cycles of visibility bands to neighbors about each bar, above and below, (resp., about each arc, outward and inward) of its visibilities to other bars (resp., arcs).

In this paper, we are interested in "exact" representations of graphs and multigraphs through bar- and arc-visibilities. Given two segments c and c' (bars in BVGs or arcs in PVGs) in a layout L, we join the corresponding vertices of a multigraph by $k \geqslant 1$ edges when in L c and c' are mutually visible in k disjoint, maximal-width bands of visibility. Specifically, given a bar-visibility layout L, we define G_L to be the plane multigraph given by replacing each bar of L by a vertex and each distinct, maximal visibility band between two bars of L by an edge, lying basically within the corresponding visibility band. Then the embedded G_L and L are *equivalent*, meaning that when the bar-rotation system of L is translated into the set of incident edge cycles, we get the rotation system of G_L. We say that a plane multigraph G is *equivalent* to a bar-visibility layout L when the plane graphs G_L and G are equivalent. The situation for the projective plane and arc-visibility is similar, but with a twist, to be discussed in Section 4.

The result that drives our proof of Theorem 1 is the next lemma. The proofs for the plane and for the projective plane are nearly identical and are similar to that of [11, Lemma 1.4.5]; a contraction edge that preserves 2-connectedness in a general graph is more easily found (see [16, p.174, Exer. 4.2.15]). The constraints for the projective planar graphs will be discussed in Section 4.

Contraction Lemma 2. *Let G be a loopless* 2-*connected graph with at least four vertices, embedded in the plane* (*respectively, on the projective plane*) *with at most one finite digon face. Then G contains an edge e such that G/e is loopless*, 2-*connected*, *and plane* (*resp.*, *projective plane*) *with at most one finite digon face*.

Proof. It suffices to find in G a simple edge e so that G/e is 2-connected since the contraction of e is done on the surface, and any unwanted multiple edges resulting from triangular faces incident with e in G are removed.

First we locate a simple edge. If the graph contains a pair of multiple edges joining, say, vertices u and v, and bounding a nonempty, contractible region (in the plane or the projective plane), then in the interior lies a block of G incident with both u and v. Since the graph has at most one digon face, that block contains at least one vertex with an incident simple edge. If G is a plane graph and satisfies the hypotheses, it might have the infinite face a digon, but when $n \geqslant 5$, at least one vertex does not lie on a digon face and is either incident with a simple edge or a pair of multiple edges bounding a nonempty, contractible region. It is routine to check that the same is true when $n = 4$. If G lies on the projective plane, then it contains a k-cycle C, $k \geqslant 4$, since G is 2-connected. Since on the projective plane every pair of noncontractible cycles must intersect, C can contain at most two consecutive edges that are multiple and each form a noncontractible 2-cycle. One additional edge of C can be multiple, forming a digon face, but all other edges of C are either simple or part of a multiple pair that bounds a nonempty, contractible region.

Next we locate a simple edge with which to contract G. Let C be a longest simple (i.e., with no repeated vertex) cycle in G; C has length at least four. C must contain a simple edge e, for otherwise, as argued above, it contains a pair of multiple edges that bound a nonempty, contractible region, containing a block incident with both endpoints, and so contradicting the maximality of C. In addition, G/e must be 2-connected, for if G/e were 1-connected, e is a chord of a simple cycle in G, contradicting the maximality of C. □

3. Bar-visibility graphs

We require one additional concept for bar and arc layouts for a subtle case [13]. Without loss of generality we may assume that a layout has bars (respectively, arcs) with distinct y-coordinates (resp., radii), but we cannot assume the same for x-coordinates (resp., θ-coordinates). Two bars (resp., arcs) in a layout are said to be *collinear* if they each have an endpoint with the same x-coordinate (resp., same θ-coordinate). Notice that a layout equivalent to a finite facial cycle of length four or more must be laid out with collinearities; see Fig. 3 for all ways to layout with bars a facial 5-cycle. Collinearities in a layout caused by bars (resp., arcs) representing vertices on a common face of length four or more are called *necessary*; all others are *unnecessary*. Notice that in Fig. 1, $K_{2,3}$ is laid out with necessary collinearities at $x = 1$ and 2, but that the unnecessary collinearities at $x = 0$ and 3 can be easily removed. So also could the unnecessary collinearities at $x = 0$, 2 in Fig. 3, unless the related vertices are involved in additional large faces, causing additional collinearities. (See [8] for a complete characterization of when collinearities must or need not occur.)

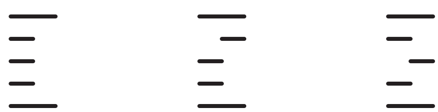

Fig. 3. All nonequivalent layouts of a 5-cycle.

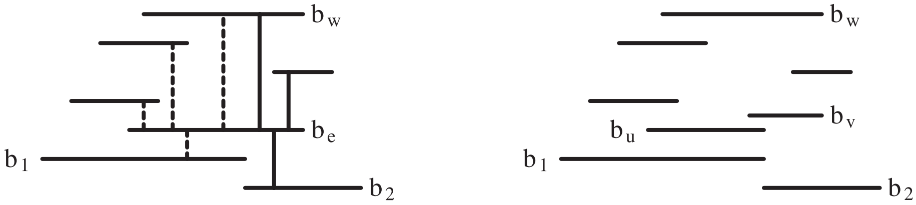

Fig. 4. The bar b_e with its u-lines and v-lines of visibility and then its replacement by b_u and b_v.

Typically for BVGs we denote bars by a, b, c and their corresponding vertices v_a, v_b, and v_c; similarly a bar for vertex v is b_v and a segment c_v. For a bar b, we denote by $y(b)$, $x_1(b)$, and $x_2(b)$, the height, the left and the right endpoint of the bar b.

Theorem 1a. *If G is a loopless, plane,* 2*-connected graph with no finite digon face and at least three vertices, then there is a bar-visibility layout L equivalent to G with no unnecessary collinearities.*

Proof. The proof is by induction on $n = |V(G)|$ and is true for $n = 3$ where G is either K_3 or K_3 with one edge doubled, the latter embedded with infinite face a digon.

Otherwise by Lemma 2, G contains an edge e so that G/e is a loopless, plane, 2-connected graph with no finite digon face. Let $e = \{u, v\}$ in G contract to new vertex u_e in G/e. By induction G/e has an equivalent bar-visibility layout L_e with only necessary collinearities and with u_e represented by bar b_e. Without loss of generality the bars of L_e lie on lines with distinct integer y-coordinates. If b_e is a bar with $y(b_e) = i$ and x-coordinate endpoints $x_1(b_e)$ and $x_2(b_e)$, then we will replace b_e by two bars at heights i and $i + 0.5$, overlapping in their x-coordinates, and spanning the interval from $x_1(b_e)$ to $x_2(b_e)$. The amount of overlap and which bar is higher is determined by the visibilities about b_e, as explained below.

Because the embeddings of G/e and L_e are equivalent and G/e is obtained by contracting the edge e in the plane, the lines of visibility from b_e to bars representing neighbors of u in G, one for each visibility band (called *u-lines*), are consecutive in the rotation of visibilities at b_e; see Fig. 4 where dashed vertical lines represent u-lines and solid vertical lines v-lines. Suppose that at the vertical line $x = p$, where visibilities along u-lines change to v-lines (called a *break point*), there is a bar b_w representing a vertex w adjacent to both u and v in G; thus $\{u, v, w\}$ forms a triangular face in G. In L_e there is a band R of visibility between b_e and b_w, and the value of p can be chosen so that the line $x = p$ lies anywhere within R. Consequently p can be chosen to avoid unnecessary collinearities. In contrast, suppose meeting $x = p$ is a bar b_1 with a u-line, but not a v-line, to b_e and a bar b_2 with a v-line, but not a u-line, to b_e. In this case a face f in G incident with $e = \{u, v\}$ has length at least four, and collinearities must occur. Thus either b_1 or b_2 must have an endpoint with x-coordinate $x = p$, and this x-coordinate is not used by any bars in L_e except those representing vertices on face f. We let $x = p$ denote the break point from u-lines to v-lines and $x = p'$ the break point from v-lines to u-lines.

Consider the u-lines and v-lines of visibility about b_e. Suppose one side of b_e contains only one type of line; for example, suppose below b_e are only u-lines. Then we may replace

b_e exactly by the bar b_u to represent vertex u, and we'll place b_v to represent v at height $i + 0.5$. The break points $p < p'$ give the left and right endpoints of b_v. Otherwise, both u-lines and v-lines include visibility to bars above and to bars below with, say, u's visibilities including the left endpoint of b_e and with v's its right endpoint. Since L_e has no unnecessary collinearities, $p \neq p'$. Then if $p < p'$, we place b_u at height i, spanning $x_1(b_e)$ to p', and b_v at height $i + 0.5$, spanning p to $x_2(b_e)$; see Fig. 4. Conversely, if $p > p'$, we place b_u at height $i + 0.5$, spanning $x_1(b_e)$ to p, and b_v at height i, spanning p' to $x_2(b_e)$. This gives a suitable visibility representation that is equivalent to the embedded graph G. □

Corollary 3. *A loopless, plane, 2-connected graph with no finite digon face and at least three vertices is a PVG.*

Proof. The bar-visibility layout of such a graph G is easily turned into one with arcs of concentric circles by "bending" a horizontal segment at height i into an arc at radius i and constructing the entire layout to span the angles of 0 to π. If the former layout is equivalent to G, then so is the latter. □

In addition it is not hard to see that a connected graph has a polar visibility layout with no visibilities through the origin if and only if the graph is a BVG [7].

4. Polar visibility graphs

In some ways it is more natural to imagine planar graphs embedded on the sphere where all faces are finite, and in this context we would allow at most one digon face, which could be punctured and opened up to form the infinite face of a planar embedding. Both depictions, on the sphere and in the plane, have the same embedding system since vertex rotations remain the same under cyclic rotation.

Graphs on the (nonorientable) projective plane involve some twists. A graph embedded on the projective plane has all faces finite, but in our PVG depiction one face will be *special*, namely the one corresponding to the infinite exterior of the layout in the plane; see Fig. 2a where this face is a 3-cycle, bounded by the arcs at radii 1, 5, and 6. This face is the equivalent of the infinite face of a plane graph and can be 2-sided; for example in Fig. 2a, if the outermost arc at radius 6 is extended to span beyond $5\pi/3$, then it and the arc at radius 5 form a digon face. In general if there is a digon face in an embedded G, in its polar visibility layout the digon will be represented by two arcs that together span more than 2π, overlap with visibility in two radial bands, and have empty exterior. (Note that two such arcs with empty interior do not form a 2-sided face.) For PVGs the relevant class of graphs then consists of the loopless graphs on the projective plane with at most one digon face. For e a simple edge of such a graph, G/e is the loopless graph obtained by contracting e on the surface, removing e, and if any digon face is thus created, replacing it by a single edge. Thus on the projective plane there may be three types of multiple edges—a pair may surround the special face, or form a contractible cycle on the surface with nonempty interior, or form a noncontractible cycle of length two.

Note that self-visibility of an arc can be achieved in a layout by an arc that spans more than π, is self-visible through the origin, and so results in a loop in a PVG; we consider this possibility for a larger class of graphs at the end of this section.

Since the projective plane is nonorientable, it is not possible to assign an orientation consistently throughout the surface. In any one depiction of a graph on this surface (with all vertices located within the representing disk), a clockwise direction can be assigned at each vertex to give a rotation system, but there are other "equivalent" representations of this embedding. In addition to the local rotations, a *signature* is needed, an assignment of ± 1 to each edge—in the disk depiction we may assign $+1$ to each edge contained wholly within the disk and -1 to each edge that crosses the boundary of the disk. Notice that if a vertex were moved across the boundary of the disk, its local rotation would be reversed as would the sign of each incident edge. More generally an embedding in any surface is equivalent to a set of local rotations of incident edges, one at each vertex, and a signature assignment to edges representing local consistency [11]. An edge $e = \{u, v\}$ is assigned $+1$ (respectively, -1) if in a small, local, contractible neighborhood of e the rotation orientation at u and v agree (resp., are reversed), meaning that both are (resp., are not) clockwise or counterclockwise. Such an assignment is called an *embedding scheme* and represents an embedding on a nonorientable surface if and only if there is a cycle in the graph with an odd number of edges with negative signature [11]. Two graphs embedded on the same surface are said to be *equivalent* if, by a series of local reversals at a vertex and its incident edges, the embedding scheme of one can be transformed into the other's.

How exactly does an embedding on the projective plane arise from a polar visibility layout L and what is the equivalent graph? As in the plane, L gives G_L^* a multigraph drawn in the plane with a vertex for each arc and an edge for each maximal-width cone joining two arcs, with edges either disjoint or now intersecting at the origin, the center of all concentric circles. To avoid edge intersection, G_L^* can and will be reembedded on the projective plane.

Lemma 4. *Given a polar visibility layout L, G_L^* is a projective planar graph that can* (*naturally*) *be embedded on the projective plane as a graph G_L so that the rotation at each vertex of G_L is the inverse of the rotation at the corresponding vertex of G_L^* and at the corresponding arc of L.*

Proof. Assume the arcs of L lie on circles of radius $1, 2, \dots, n$ where $n = |V(G)|$. This naturally leads to another layout in a disk D of radius $n + 1$ and centered at the origin by inverting each circle and arc in L through the circle of radius $(n + 1)/2$. That is, each point of L with polar coordinates (r, θ), $0 < r < n + 1$, is mapped by the inversion to the point $(n+1-r, \theta)$ in D. This inversion preserves circles, arcs, and the angles defining these arcs. We denote the inverted layout of L by $i(L)$; see Fig. 2a,b.

Identifying opposite points of the boundary of D, we create a projective plane. Two arcs in $i(L)$ that were previously radially visible in a cone that did not contain the origin are still radially visible in $i(L)$, and a pair visible in a cone through the origin are now visible in a "generalized cone" that crosses the boundary of the projective plane, reemerging on the other side. The coordinates of such a generalized cone are given by $\{(r, \theta) : r^* \leqslant r \leqslant n+1 \textit{ or } -(n+1) \leqslant r \leqslant -s^*;\ \theta_1 \leqslant \theta \leqslant \theta_2\}$, where $r^*, s^*, \theta_1 < \theta_2$ are constants, $0 \leqslant r^*, s^* < n+1$. In addition, the interiors of no two of these new cones intersect, and as with bar-visibility

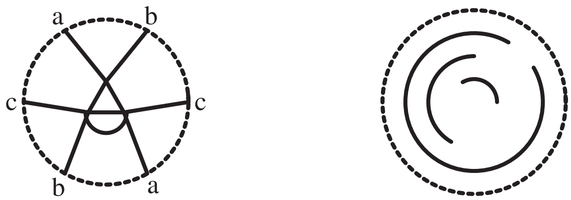

Fig. 5. K_3 with multiple edges on the projective plane and the equivalent polar visibility layout.

layouts, $i(L)$ gives rise to a graph G_L embedded on the projective plane without crossing edges.

The inversion reverses the cycle of visibility bands about an arc of L to that of the corresponding arc of $i(L)$ so that the rotation system for G_L and for L consist of inverses at each corresponding vertex-arc pair. Since the rotation cycles for L and for G_L^* agree, the lemma follows. $\square$

Then we say G_L, embedded on the projective plane, and L in the plane are *equivalent* because the embedding rotation systems are inverses of each other and the edge-signatures agree. More generally we say that a projective plane multigraph G is *equivalent* to a polar visibility layout L when the projective plane graphs G_L and G are equivalent in their embedding schemes.

Theorem 1b. *If G is a loopless, projective plane,* 2*-connected graph with at most one digon face and with at least three vertices, then there is a polar visibility layout L equivalent to G with no unnecessary collinearities.*

The proof, by induction on $n = |V(G)|$, is identical to that of Theorem 1a except for the base case and for the use of the projective planar version of Lemma 2. On the projective plane for $n = 3$, K_3 with any edges doubled and possibly one edge tripled has an embedding with at most one digon face and an equivalent PVG layout as in Fig. 5. So also do all subgraphs of this graph; note that the subgraphs embedded with no noncontractible cycle, simple K_3 or with one edge doubled, have equivalent bar-visibility layouts, and by Corollary 3 have equivalent polar visibility layouts. For $n > 3$, we apply Lemma 2, and the proof of Theorem 1a can be followed exactly, except for the obvious substitutions of arc, radii, θ-coordinates, etc.

By using long arcs, ones that span more than π radians, the same technique yields the following.

Corollary 5. *If G is a projective plane,* 2*-connected graph with at least three vertices, embedded with at most one digon face and with one noncontractible loop, then there is a polar visibility layout equivalent to G.*

5. Related algorithms and concluding remarks

There are well-known linear-time algorithms for recognizing and laying out bar-visibility graphs [12,14,18]. The background algorithms for determining whether a graph is planar, 2-connected, or embeddable with cut-vertices on a common face are also well studied and linear; see, for example, [4]. There is also a linear algorithm for detecting and embedding projective planar graphs [9]. The condition that a graph embed with at most one digon face can also be quickly determined, by examining the multiple edges and the resulting cutsets of two vertices, though this may require $O(e^2) = O(n^2)$ time since $e = O(n)$ for graphs in the plane or projective plane.

The new inductive proof of Theorem 1 leads to recursive algorithms for the layout of these classes of 2-connected plane and projective plane multigraphs; however, the running time for a straightforward implementation appears to be $O(n^3)$, with suitable data structures and storage. Thus the recursive algorithm is not competitive with that for BVGs, but is currently the only known approach for laying out PVGs.

Acknowledgements

The author would like to thank Therese Biedl, Alice Dean, Joe O'Rourke, Tom Shermer, Carsten Thomassen, and Stan Wagon for help and encouragement with this approach.

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